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Unphysical Predictions of Some Doubly Special Relativity Theories

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A kind of doubly special relativity theory proposed by J. Magueijo and L. Smolin [Phys. Rev. Lett. **88**, 190403 (2002)] is analysed. It is shown that this theory leads to serious physical difficulties in interpretation of kinematical quantities. Moreover, it is argued that statistical mechanics and thermodynamics cannot be resonably formulate within the model proposed in the mentioned paper.

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Recently J. Magueijo and L. Smolin [1] have proposed a modification of special relativity with intrinsically built in Planck length. Their construction is based on a non-linear form of the Lorentz group action in the momentum space. This is related to a special choice of the Lie algebra of the Lorentz group in the enveloping algebra of the nilpotent Lie algebra generated by $\partial/\partial p_\mu$, p_μ and I. Accordingly to [1] the Lorentz boosts in the x-direction read

$$p_0' = \frac{p_0\cosh\xi + p_1\sinh\xi}{1 + lp_0(\cosh\xi - 1) + lp_1\sinh\xi}, \tag{1a}$$

$$p_1' = \frac{p_1\cosh\xi + p_0\sinh\xi}{1 + lp_0(\cosh\xi - 1) + lp_1\sinh\xi}, \tag{1b}$$

$$p_2' = \frac{p_2}{1 + lp_0(\cosh\xi - 1) + lp_1\sinh\xi}, \tag{1c}$$

$$p_3' = \frac{p_3}{1 + lp_0(\cosh\xi - 1) + lp_1\sinh\xi}, \tag{1d}$$

where l is interpreted as the Planck length l_P. The above transformations together with rotations close to the Lorentz group and leave unchanged the following invariant built from the particle four-momentum p_μ:

$$m^2c^2 = \frac{\eta^{\mu\nu}p_\mu p_\nu}{(1 - lp_0)^2}, \tag{2}$$

when c, as usually, denotes the speed of light.

The scheme proposed in [1] belongs to the class of recently investigated models known as "doubly special relativity" [1, 2, 3, 4, 5, 6, 7, 8]. In this paper we analyze this scheme in details and we show that it is plagued by physically unacceptable features.

To present our discussion as simply as possible, in the following we restrict ourselves to the two-dimensional case. We will denote the energy and momentum as cp_0 and p, respectively. Furthermore, to separate the dimensional and scale parameters of the model we use the quantity λ/m_Pc instead of l, where λ is dimensionless and $m_P = \sqrt{\hbar c/G_N}$ is the Planck mass. It is enough to consider positive values of λ. Therefore the Lorentz transformations (1) take the form

$$p_0' = \frac{p_0\cosh\xi + p\sinh\xi}{1 + \frac{\lambda}{m_Pc}p_0(\cosh\xi - 1) + \frac{\lambda}{m_Pc}p\sinh\xi}, \tag{3a}$$

$$p' = \frac{p_0\sinh\xi + p\cosh\xi}{1 + \frac{\lambda}{m_Pc}p_0(\cosh\xi - 1) + \frac{\lambda}{m_Pc}p\sinh\xi}, \tag{3b}$$

while the invariant (2) reads

$$\frac{p_0^2 - p^2}{\left(1 - \frac{\lambda}{m_Pc}p_0\right)^2} = m^2c^2. \tag{4}$$

Let us consider firstly the possible solutions of (4). It follows that we have singularity at $p_0 = m_Pc/\lambda$, so possible range of p_0 is restricted to the two distinct areas: $p_0 < m_Pc/\lambda$ or $p_0 > m_Pc/\lambda$.

Now, for $p_0 \neq m_Pc/\lambda$ Eq. (4) determines algebraic conics; namely

hyperbola,	when $m < m_P/\lambda$,	(5a)
parabola,	when $m = m_P/\lambda$,	(5b)
ellipse,	when $m > m_P/\lambda$,	(5c)
the pair of half-lines,	when $m = 0$.	(5d)

Under physical condition that for the free motion the energy cp_0 is bounded from below we obtain four possible situations showed in the Fig. 1. The line $p_0 = m_Pc/\lambda$ divides each curve into distinct parts: upper and lower ones. The momentum manifold must be an orbit of the Lorentz group realized as in (3). We can easily prove that this is possible only for the lower parts of conics showed in the Fig. 1(a,b,d); on the upper parts of these conics the Lorentz group cannot be globally realized. Moreover, the upper part of the curve showed in the Fig. 1(c), despite of the fact that it forms an orbit of the Lorentz group, is also unphysical because the energy of the particle reaches the maximum when its momentum vanishes. Furthermore, if we restricts ourselves to these physically acceptable parts, denoted in the Fig. 1 by the solid lines, we can linearize the transformations (3) by the following map [6]

$$k_\mu = \frac{p_\mu}{1 - \frac{\lambda}{m_Pc}p_0}, \tag{6}$$

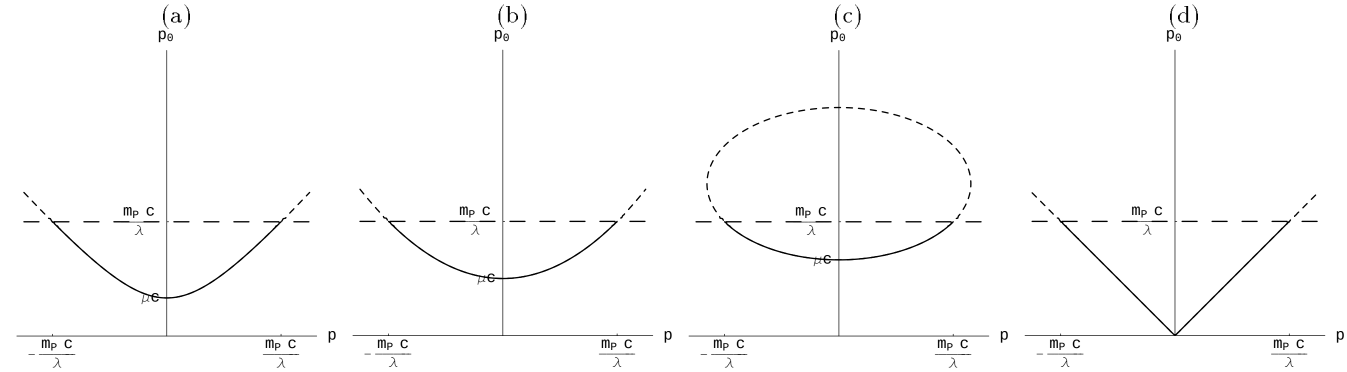

FIG. 1: Possible positive solutions of the dispersion relation (4) for: (a) $m < m_P/\lambda$, (b) $m = m_P/\lambda$, (c) $m > m_P/\lambda$, (d) $m = 0$; $\mu = m/(1 + \lambda m/m_P)$ is the mass of the particle (i.e. the minimum of the energy). The momentum p and energy cp_0 are bounded: $-m_P c/\lambda < p < m_p c/\lambda$, $\mu c \leq p_0 < m_P c/\lambda$

where the momentum components k_μ form the usual Minkowski vector. Note, that the Jacobian determinant of this transformation vanishes only for $p_0 = m_P c/\lambda$. under the action of the Lorentz group k_μ's transform linearly

$$k_0' = k_0 \cosh\xi + k \sinh\xi, \tag{7a}$$

$$k' = k \cosh\xi + k_0 \sinh\xi, \tag{7b}$$

and leaves the invariant $k_0^2 - k^2 = m^2c^2$ unchanged. Note, that the minima of p_0 and k_0 (i.e. the masses) are related by $\mu = m/(1 + \lambda m/m_P)$.

Concluding the above discussion: The nonlinear transformation law (3) is a consequence of the choice of a nonlinear coordinate system in the momentum space and it is not *essentially nonlinear realization of the Lorentz group* [13], because it can be linearized by the appropriate choice of coordinates (6).

The question if the momentum coordinates p_0 and p are physically admissible is open. In particular we do not have the notion of the inertial frame (observer) defined operationally. Therefore, up to now, we cannot properly answer this question (in contrast to the statement in e.g. [1, 9]). However, if we agree that canonical formalism can be used in such a simple kinematical problem (i.e. the free motion), we can try to partially answer this question. According to [1, 9] let us identify the energy with the generator of translation in time (Hamiltonian), i.e. $H = cp_0$, while p with the canonical momentum. Therefore the particle velocity v can be calculated from one of the Hamilton equations

$$v = \frac{\partial H}{\partial p} = \frac{pc}{p_0\left(1 - \frac{\lambda^2 m^2}{m_P^2}\right) + \frac{\lambda^2 m^2 c}{m_P}}. \tag{8}$$

Now, with help of (3), (4) and (8) we are able to find the transformation law for the particle velocity under the action of the Lorentz group

$$v' = \frac{c\left[\frac{v}{c}\left(\cosh\xi + \frac{v}{c}\sinh\xi\right) + \frac{m_P}{\lambda m}\sqrt{1 - \frac{v^2}{c^2}\left(1 - \frac{\lambda^2 m^2}{m_P^2}\right)}\left(\frac{v}{c}\cosh\xi + \sinh\xi\right)\right]}{1 - \frac{v^2}{c^2} + \frac{v}{c}\left(\frac{v}{c}\cosh\xi + \sinh\xi\right) + \frac{m_P}{\lambda m}\sqrt{1 - \frac{v^2}{c^2}\left(1 - \frac{\lambda^2 m^2}{m_P^2}\right)}\left(\cosh\xi + \frac{v}{c}\sinh\xi\right)}. \tag{9}$$

The transformation law (9) goes to the standard Lorentz transformation law for velocities when $\lambda \to 0$ or when $m = 0$ ($\mu = 0$) and $v = c$. However, for $\lambda \neq 0$ and $m \neq 0$ it depends on the particle mass μ (recall that $m = \mu/(1 - \lambda\mu/m_P)$). Such a situation is obviously in conflict with our physical space-time intuition: Indeed, let us imagine two bodies A and B with mass m_A and m_B, respectively, moving in an inertial frame with the same velocity v (for simplicity one can assume $v = 0$). From the point of view of Lorentz boosted observer (i.e. from another inertial frame), if $m_A \neq m_B$ and $\lambda \neq 0$, these two bodies *have different velocities*! This is very undesirable feature from the physical point of view.

To understand better this issue let us look for the relationship of v and the Lorentz velocity (i.e. the velocity which appears in the usual, *linear*, Lorentz transformations) $v_L = ck/k_0$. By means of (6) and (8) we get that

$$v = \frac{v_L}{1 + \frac{\lambda m}{m_P}\sqrt{1 - \frac{v_L^2}{c^2}}}. \tag{10}$$

The velocity v_L has the proper physical interpretation as the velocity canonically related to k, i.e. $v_L = \partial k_0/\partial k = k/k_0$. For the other hand it is defined as $v_L = dx/dt$, where t and x are usual Minkowskian time and coordinate. However, v_L cannot be canonically related to the momentum p. Notice that $v_L = k/k_0 = p/p_0$ and it has the standard transformation law.

Concluding, we have serious problems with the velocity transformation law and with the definition of the inertial observers within the model presented in [1].

The next difficulty occurs when we try to formulate statistical mechanics and thermodynamics within the framework of theory proposed by Magueijo and Smolin [1]. Let us return to the three-dimensional case. It is easy to see that the invariant measure in the momentum space is given by the following formula

$$d\Gamma = \frac{\left(1 - \frac{\lambda^2 m^2}{m_P^2}\right)^3 d^3p}{2\left[1 - \frac{\lambda}{m_P c}\sqrt{m^2c^2 + |\vec{p}|^2\left(1 - \frac{\lambda^2 m^2}{m_P^2}\right)}\right]^3 \sqrt{m^2c^2 + |\vec{p}|^2\left(1 - \frac{\lambda^2 m^2}{m_P^2}\right)}}, \tag{11}$$

(this holds also for the case $m = \mu = 0$) and, since the momentum space is bounded (see Fig. 1), the one-particle partition function is

$$Z_1 = V \iiint_{-\frac{m_P c}{\lambda}}^{\frac{m_P c}{\lambda}} d\Gamma\, e^{-\beta E}. \tag{12}$$

But the measure (11) is singular at $|\vec{p}| \to m_P c/\lambda$ and the energy approaches the limit $m_P c^2/\lambda$ when $|\vec{p}| \to m_P c/\lambda$, so the integral (12) is divergent. Consequently the partition function does not exist in the considered case, as well as the internal energy and the entropy. The same holds in arbitrary N-particle case.

If we take non-relativistic measure in the momentum space d^3p instead of $d\Gamma$, the integral (12) becomes convergent, but on the other hand the energy of the free gas of N identical particles (either massive or massless) is equal to [9]

$$E = m_P c^2 \frac{\sum_{i=1}^{N} \frac{E_i}{m_P c^2 - \lambda E_i}}{1 + \sum_{i=1}^{N} \frac{\lambda E_i}{m_P c^2 - \lambda E_i}} \tag{13}$$

and

$$E \approx \frac{m_P c^2}{\lambda} \tag{14}$$

for large N [14]. The partition function Z of the N-particle free gas is then of the form

$$Z \approx e^{-\beta \frac{m_P c^2}{\lambda}} \left[8\left(\frac{m_P c}{\lambda}\right)^3 V\right]^N \tag{15}$$

and, consequently, we come at the curious conjecture that the internal energy in the thermodynamical limit (i.e. for $N \to \infty$)

$$U = -\frac{\partial \ln Z}{\partial \beta} = \frac{m_P c^2}{\lambda} \tag{16}$$

does not depend on temperature!

Concluding this point of the discussion we have to state that statistical mechanics and/or thermodynamics do not exist within the model presented in [1].

We would like to point out that the non-additivity of energy seems to be a common feature of doubly special relativity theories (see [9]). Therefore, the question of the possiblity of formulation of statistical mechanics or thermodynamics in such theories is still open [15].

We can conclude that the model proposed by Magueijo and Smolin [1] encounters some serious difficulties discussed above. First of all, they are interpretational problems of space-time quantities: coordinates, velocities, etc. Moreover, it seems to be impossible to formulate resonably statistical mechanics and thermodynamics within this framework.

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[1] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).
[2] G. Amelino-Camelia and T. Piran, Phys. Rev. D **64**, 036005 (2001).
[3] G. Amelino-Camelia, Phys. Lett. B **510**, 255 (2001).
[4] G. Amelino-Camelia, in *New Developments in Fundamental Interaction Theories*, edited by J. Lukierski and

J. Rembieliński (American Institute of Physics, Melville, NY, 2001), vol. 589 of *AIP Conference Proceedings*, pp. 137–150.
[5] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002).
[6] J. Lukierski and A. Nowicki, arXiv:hep-th/0203065.
[7] J. Kowalski-Glikman and S. Nowak, arXiv:hep-th/0203040.
[8] J. Kowalski-Glikman and S. Nowak, arXiv:hep-th/0204245.
[9] S. Judes and M. Visser, arXiv:gr-qc/0205067.
[10] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969).
[11] C. G. Callan, Jr., S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).
[12] J. Kowalski-Glikman, arXiv:hep-th/0111101.
[13] Strictly speaking, the nonlinearity is an illusion in this case: As it is well known [10, 11] essentially nonlinear group realizations are connected with the group action in the homogeneous coset spaces G/H, where H is a subgroup of the group G.
[14] Note, that for the choice $\lambda = 1$, as it is assumed in [1], the mass of any large enough system is nearly equal to the Planck mass, i.e. to 2.176×10^{-8} kg. On the other hand, there is no other natural physical choice for the parameter λ.
[15] In the paper [12] only the one-particle partition function for the massless case is discussed within the model proposed in [2, 3, 4, 5].